

Claremont Colleges Scholarship @ Claremont

CMC Faculty Publications and Research

CMC Faculty Scholarship

1-1-2016

On an Effective Variation of Kronecker's Approximation Theorem

Lenny Fukshansky
Claremont McKenna College

Recommended Citation

Fukshansky, Lenny, On an effective variation of Kronecker's approximation theorem, preprint.

This Article - preprint is brought to you for free and open access by the CMC Faculty Scholarship at Scholarship @ Claremont. It has been accepted for inclusion in CMC Faculty Publications and Research by an authorized administrator of Scholarship @ Claremont. For more information, please contact scholarship@cuc.claremont.edu.

ON AN EFFECTIVE VARIATION OF KRONECKER'S APPROXIMATION THEOREM

LENNY FUKSHANSKY

ABSTRACT. Let $\Lambda \subset \mathbb{R}^n$ be an algebraic lattice, coming from a projective module over the ring of integers of a number field K . Let $Z \subset \mathbb{R}^n$ be the zero locus of a finite collection of polynomials such that $\Lambda \not\subset Z$ or a finite union of proper full-rank sublattices of Λ . Let K_1 be the number field generated over K by coordinates of vectors in Λ , and let L_1, \dots, L_t be linear forms in n variables with algebraic coefficients satisfying an appropriate linear independence condition over K_1 . For each $\varepsilon > 0$ and $\mathbf{a} \in \mathbb{R}^n$, we prove the existence of a vector $\mathbf{x} \in \Lambda \setminus Z$ of explicitly bounded sup-norm such that

$$\|L_i(\mathbf{x}) - a_i\| < \varepsilon$$

for each $1 \leq i \leq t$, where $\|\cdot\|$ stands for the distance to the nearest integer. The bound on sup-norm of \mathbf{x} depends on Λ , K , Z , heights of linear forms, and ε . This presents a generalization of Kronecker's approximation theorem, establishing an effective result on density of the image of $\Lambda \setminus Z$ under the linear forms L_1, \dots, L_t in the t -torus $\mathbb{R}^t/\mathbb{Z}^t$.

1. INTRODUCTION

Let $1, \theta_1, \dots, \theta_t$ be \mathbb{Q} -linearly independent real numbers. The classical approximation theorem of Kronecker then states that the set of points

$$\{(\{n\theta_1\}, \dots, \{n\theta_t\}) : n \in \mathbb{Z}\}$$

is dense in the t -torus $\mathbb{R}^t/\mathbb{Z}^t$, where $\{\cdot\}$ stands for the fractional part of a real number. This result was originally obtained by Kronecker [13] in 1884, and presents a deep generalization of Dirichlet's 1842 theorem on Diophantine approximation [5]; see, for instance, [11] for a detailed exposition of these classical results.

Kronecker's theorem can also be viewed as a statement on density of the image of the integer lattice under collection of linear forms in the torus $\mathbb{R}^t/\mathbb{Z}^t$ (compare to the famous Oppenheim conjecture for quadratic forms). Specifically, if L_1, \dots, L_t are linear forms in n variables with real coefficients b_{ij} so that the set of numbers 1 and b_{ij} are linearly independent over \mathbb{Q} , then for any $\varepsilon > 0$ and $\mathbf{a} \in \mathbb{R}^t$ there exists $\mathbf{x} \in \mathbb{Z}^n$ such that

$$(1) \quad \|L_i(\mathbf{x}) - a_i\| < \varepsilon \quad \forall 1 \leq i \leq t,$$

where $\|\cdot\|$ stands for the distance to the nearest integer. A nice survey of a wide variety of results related to Kronecker's theorem is given in [9]. There are also some effective versions of Kronecker's theorem, where the size of the coordinates of the vector \mathbf{x} in (1) are bounded, under some additional algebraic assumptions on the

2010 *Mathematics Subject Classification*. 11H06, 11G50, 11J68, 11D99.

Key words and phrases. Kronecker's theorem, Diophantine approximation, heights, polynomials, lattices.

The author was partially supported by the NSA grant H98230-1510051.

coefficients of the linear forms: see, for instance, [15], [17]. The main goal of this note is to extend the previously known effective versions of Kronecker's theorem in two ways:

- (1) allow for the approximating vector \mathbf{x} as in the equation (1) above to come from a more general lattice than \mathbb{Z}^n ,
- (2) exclude vectors from any prescribed union of projective varieties or sublattices not containing this lattice.

To give precise statements of our results, we need some notation.

1. *The lattice.* Let $n \geq 1$ be an integer, and for each vector $\mathbf{x} \in \mathbb{R}^n$ define the sup-norm

$$|\mathbf{x}| := \max_{1 \leq i \leq n} |x_i|.$$

Let K be a number field of degree $d = r_1 + 2r_2$ over \mathbb{Q} , where r_1 and r_2 are numbers of its real and complex places, respectively, and write \mathcal{O}_K for its ring of integers. Let $1 \leq s \leq w$ be integers, and let $\mathcal{M} \subset K^w$ be an \mathcal{O}_K -module such that $\mathcal{M} \otimes_K K \cong K^s$. Write $\mathcal{D}_K(\mathcal{M})$ for the discriminant of \mathcal{M} . Define $\mathfrak{U}_K(\mathcal{M})$, a fractional \mathcal{O}_K -ideal in K , to be

$$(2) \quad \mathfrak{U}_K(\mathcal{M}) = \{\alpha \in K : \alpha \mathcal{M} \subseteq \mathcal{O}_K^w\}.$$

We let $\Lambda_K(\mathcal{M}) \subset \mathbb{R}^{wd}$ be the lattice of rank sd , which is the image of \mathcal{M} under the standard Minkowski embedding.

2. *The projective varieties.* Let $m \geq 1$ be an integer. For each $1 \leq i \leq m$, let \mathcal{S}_i be a finite set of homogeneous polynomials in $\mathbb{R}[x_1, \dots, x_{wd}]$ and $Z(\mathcal{S}_i)$ be its zero set in \mathbb{R}^{wd} , that is,

$$Z(\mathcal{S}_i) = \{\mathbf{x} \in \mathbb{R}^{wd} : P(\mathbf{x}) = 0 \text{ for all } P \in \mathcal{S}_i\}.$$

For the collection $\mathcal{S} := \{\mathcal{S}_1, \dots, \mathcal{S}_m\}$ of finite sets of homogeneous polynomials, define

$$(3) \quad \mathcal{Z}_{\mathcal{S}} := \bigcup_{i=1}^m Z(\mathcal{S}_i),$$

and

$$(4) \quad M_{\mathcal{S}} := \sum_{i=1}^m \max\{\deg P : P \in \mathcal{S}_i\}.$$

We allow for the possibility that $\mathcal{Z}_{\mathcal{S}} = \{\mathbf{0}\}$, in which case we take instead $M_{\mathcal{S}} = 1$. Notice that $\mathcal{Z}_{\mathcal{S}}$ is an algebraic set, which is a union of a finite collection of projective varieties. Assume that the lattice $\Lambda_K(\mathcal{M})$ is not contained in the set $\mathcal{Z}_{\mathcal{S}}$.

3. *The linear forms.* Let $K_1 = K(\Lambda_K(\mathcal{M}))$, i.e. K_1 is the number field generated over K by the entries of any basis matrix of the lattice $\Lambda_K(\mathcal{M})$. Let $B := (b_{ij})_{1 \leq i \leq t, 1 \leq j \leq wd}$ be a $t \times wd$ matrix with real algebraic entries so that $1, b_{11}, \dots, b_{t(wd)}$ are linearly independent over K_1 , and let $\ell = [E : \mathbb{Q}]$ where $E = K_1(b_{11}, \dots, b_{t(wd)})$. We will also write $\ell_v = [E_v : \mathbb{Q}_v]$ for the local degree of E at every place $v \in M(E)$. Define t linear forms in wd variables

$$(5) \quad L_i(x_1, \dots, x_{wd}) = \sum_{j=1}^{wd} b_{ij} x_j \in \mathbb{R}[x_1, \dots, x_{wd}] \quad \forall 1 \leq i \leq t.$$

Our first goal here is to prove the following effective result on density of the image of the set $\Lambda_K(\mathcal{M}) \setminus \mathcal{Z}_S$ under the linear forms L_1, \dots, L_t in the torus $\mathbb{R}^t/\mathbb{Z}^t$. Let h denote the usual Weil height on algebraic numbers, as well as its extension to vectors with algebraic coordinates; we recall the definition of height along with other necessary notation in Section 2.

Theorem 1.1. *Let $\mathbf{a} = (a_1, \dots, a_t) \in \mathbb{R}^t$ and $\varepsilon > 0$. There exist $\mathbf{x} \in \Lambda_K(\mathcal{M}) \setminus \mathcal{Z}_S$ and $\mathbf{p} \in \mathbb{Z}^t$ such that*

$$|L_i(\mathbf{x}) - a_i - p_i| < \varepsilon$$

and

$$|\mathbf{x}| \leq \mathbf{a}_K(t, \ell, s) (sdM_S |\mathcal{D}_K(\mathcal{M})|^{\frac{s}{2}})^{\mathfrak{K}+1} \left((wd)^{\frac{3}{2}} h(B) \right)^{\mathfrak{K}} \mathbf{c}_K(\mathcal{M}, \ell, t) \varepsilon^{-\ell+1},$$

where the exponent $\mathfrak{K} = \ell^2(t+1) - \ell$ and the constants are

$$\mathbf{a}_K(t, \ell, s) = 2^{\ell t(\ell-1) + sr_1 \mathfrak{K} + \frac{sd-1}{2}} (t+1)^{3\ell-1} (t!)^{2\ell}$$

and

$$\mathbf{c}_K(\mathcal{M}, \ell, t) = \min \left\{ h(\alpha)^{(\mathfrak{K}+1)sd-1} h(\alpha^{-1})^{\mathfrak{K}} : \alpha \in \mathfrak{U}_K(\mathcal{M}) \right\}.$$

One special case of Theorem 1.1 is when \mathcal{Z}_S is a union of linear spaces, which means that the point \mathbf{x} in question is in $\Lambda_K(\mathcal{M})$ but outside of a union of sublattices of smaller rank than $\Lambda_K(\mathcal{M})$. What if the rank of such sublattices is equal to the rank of $\Lambda_K(\mathcal{M})$? The next theorem addresses this situation.

Theorem 1.2. *Let $\mathbf{a} = (a_1, \dots, a_t) \in \mathbb{R}^t$ and $\varepsilon > 0$. Let $m > 0$ and $\Gamma_1, \dots, \Gamma_m \subset \Lambda_K(\mathcal{M})$ be proper sublattices of full rank and respective determinants $\mathcal{D}_1, \dots, \mathcal{D}_m$, and let $\mathcal{D} = \mathcal{D}_1 \cdots \mathcal{D}_m$. Then for every $\alpha \in \mathfrak{U}_K(\mathcal{M})$ there exist $\mathbf{x} \in \Lambda_K(\mathcal{M}) \setminus \bigcup_{i=1}^m \Gamma_i$ and $\mathbf{p} \in \mathbb{Z}^t$ such that*

$$|L_i(\mathbf{x}) - a_i - p_i| < \varepsilon$$

and

$$|\mathbf{x}| \leq \left(\mathbf{b}_K(t, \ell, s, w) (h(\alpha)h(\alpha^{-1})h(B)\mathcal{E}_\alpha(\mathcal{M}, \Gamma_1, \dots, \Gamma_m))^{\mathfrak{K}} \frac{\mathcal{D} \varepsilon^{-\ell+1}}{|\mathcal{D}_K(\mathcal{M})|^{\frac{sm}{2}}} + 1 \right) \mathcal{E}_\alpha,$$

where the exponent $\mathfrak{K} = \ell^2(t+1) - \ell$, as in Theorem 1.1, the constant

$$\mathbf{b}_K(t, \ell, s, w) = 2^{\ell t(\ell-1) + \frac{\mathfrak{K}}{2} + smr_2} (t+1)^{3\ell-1} (t!)^{2\ell} (wd)^{\frac{3\mathfrak{K}}{2}},$$

and $\mathcal{E}_\alpha =$

$$(6) \quad \mathcal{E}_\alpha(\mathcal{M}, \Gamma_1, \dots, \Gamma_m) := 2^{\frac{sr_1-1}{2}} h(\alpha)^{sd-1} |\mathcal{D}_K(\mathcal{M})|^{\frac{s}{2}} \left(\sum_{i=1}^m \frac{\mathcal{D}}{\mathcal{D}_i} - m + 1 \right) + \mathcal{D}^{\frac{1}{sd}}.$$

Here is a sketch of the proofs of Theorems 1.1 and 1.2. We first construct a point $\mathbf{y} \in \Lambda_K(\mathcal{M})$ of controlled sup-norm, which is outside of \mathcal{Z}_S or $\bigcup_{i=1}^m \Gamma_i$, respectively: in the first case, we use the classical Minkowski's Successive Minima Theorem and a version of Alon's Combinatorial Nullstellensatz [1] (we use the convenient formulation developed in [7]), while in the second we employ a recent result of Henk and Thiel [12] on points of small norm in a lattice outside of a union of full-rank sublattices. We use \mathbf{y} to construct an infinite sequence of points $n\mathbf{y}$ satisfying the above conditions, and use an effective version of Kronecker's original theorem to obtain a value of the index n (depending on $\varepsilon > 0$) for which the required inequalities on values of linear forms are satisfied. A convenient effective

version of Kronecker's theorem that we use is worked out in Section 3. It should be remarked that the most important feature of approximation results such as our Theorems 1.1 and 1.2 is the exponent on ε in the bounds for $|\mathbf{x}|$. As we show, this exponent is the same as in the corresponding bound of the effective version of Kronecker's theorem that we use.

Remark 1.1. As mentioned above, there are some effective versions of Kronecker's theorem in the literature, such as the one produced by Malajovich [15]. Vesselin Dimitrov also informed me that a similar result can also be obtained using the Erdős-Turan-Koksma inequality in discrepancy theory (see Theorem 1.21 of [6]). The version derived in Section 3 however has a substantially better exponent on ε ; this argument, based on Liouville's inequality and a homogeneous-inhomogeneous transference principle was suggested by the referee.

In Section 2 we introduce the necessary notation and provide all the details of our setup. We then prove Theorem 1.1 in Section 4 and Theorem 1.2 in Section 5.

2. NOTATION AND SETUP

Let the notation be as in Section 1. Here we introduce some additional notation needed for our algebraic setup. Let the number field K have discriminant \mathcal{D}_K , r_1 real embeddings $\sigma_1, \dots, \sigma_{r_1}$ of K , and r_2 conjugate pairs of complex embeddings $\tau_1, \bar{\tau}_1, \dots, \tau_{r_2}, \bar{\tau}_{r_2}$, then $d = r_1 + 2r_2$. For each τ_k , write $\Re(\tau_k)$ for its real part and $\Im(\tau_k)$ for its imaginary part. Let us write $M(K)$ for the set of all places of K , then the archimedean places of K are in correspondence with the embeddings of K , and we choose the absolute values $|\cdot|_{v_1}, \dots, |\cdot|_{v_{r_1+r_2}}$ so that for each $a \in K$

$$|a|_{v_k} = |\sigma_k(a)| \quad \forall 1 \leq k \leq r_1$$

and

$$|a|_{v_{r_1+k}} = |\tau_k(a)| = \sqrt{\Re(\tau_k(a))^2 + \Im(\tau_k(a))^2} \quad \forall 1 \leq k \leq r_2,$$

where $|\cdot|$ stands for the usual absolute value on \mathbb{R} or \mathbb{C} , respectively. For each $v \in M(K)$, we write K_v for the completion of K at v , and for each $n \geq 1$ we define a local norm $|\cdot|_v : K_v^n \rightarrow \mathbb{R}$ by

$$|\mathbf{a}|_v := \max_{1 \leq j \leq n} |a_j|_v,$$

for each $\mathbf{a} = (a_1, \dots, a_n) \in K_v^n$. Then the extended Weil height on K^n is given by

$$h(\mathbf{a}) = \prod_{v \in M(K)} \max\{1, |\mathbf{a}|_v\}^{d_v/d},$$

where $d_v = [K_v : \mathbb{Q}_v]$ is the local degree of K at v , so that $\sum_{v|u} d_v = d$ for each $u \in M(\mathbb{Q})$.

For each integer $n \geq 1$, define the standard Minkowski embedding $\rho_K^n : K^n \rightarrow \mathbb{R}^{nd}$ by

$$\rho_K^n(\mathbf{a}) := (\sigma_1^n(\mathbf{a}), \dots, \sigma_{r_1}^n(\mathbf{a}), \Re(\tau_1^n(\mathbf{a})), \Im(\tau_1^n(\mathbf{a})), \dots, \Re(\tau_{r_2}^n(\mathbf{a})), \Im(\tau_{r_2}^n(\mathbf{a}))).$$

We will now use Minkowski embedding to construct lattices from \mathcal{O}_K -modules and outline some of their main properties; see [8] for further details. Let $1 \leq s \leq w$ be integers, and let $\mathcal{M} \subset K^w$ be an \mathcal{O}_K -module such that $\mathcal{M} \otimes_K K \cong K^s$. By the

structure theorem for finitely generated projective modules over Dedekind domains (see, for instance [14]),

$$\mathcal{M} = \left\{ \sum_{j=1}^s \beta_j \mathbf{y}_j : \mathbf{y}_j \in \mathcal{O}_K^w, \beta_j \in \mathcal{I}_j \right\}$$

for some \mathcal{O}_K -fractional ideals $\mathcal{I}_1, \dots, \mathcal{I}_s$ in K . By Proposition 13 on p.66 of [14], the discriminant of \mathcal{M} is then

$$(7) \quad \mathcal{D}_K(\mathcal{M}) := \mathcal{D}_K \prod_{j=1}^s \mathbb{N}(\mathcal{I}_j)^2,$$

where $\mathbb{N}(\mathcal{I}_j)$ is the norm of the fractional ideal \mathcal{I}_j .

Let $\Lambda_K(\mathcal{M}) := \rho_K^w(\mathcal{M})$ be an algebraic lattice of rank sd in \mathbb{R}^{wd} , then a direct adaptation of Lemma 2 on p.115 of [14] implies that the determinant of $\Lambda_K(\mathcal{M})$ is

$$(8) \quad \det(\Lambda_K(\mathcal{M})) = 2^{-sr_2} |\mathcal{D}_K(\mathcal{M})|^{\frac{s}{2}} = 2^{-sr_2} |\mathcal{D}_K|^{\frac{s}{2}} \prod_{j=1}^s \mathbb{N}(\mathcal{I}_j),$$

where the last identity follows by (7) above. Let $\mathbf{x} \in \Lambda_K(\mathcal{M})$, then $\mathbf{x} = \rho_K^w(\mathbf{a})$ for some $\mathbf{a} \in \mathcal{M}$ and

$$(9) \quad |\mathbf{x}| \geq \frac{1}{\sqrt{2}} h(\alpha)^{-1},$$

for any $\alpha \in \mathfrak{U}_K(\mathcal{M})$ by inequality (54) of [8]. Let $v \in M(K)$ be an archimedean place, and assume first that it corresponds to a real embedding σ_j for some $1 \leq j \leq r_1$, then $|\mathbf{a}|_v = |\mathbf{x}|$. On the other hand, if v corresponds to a complex embedding τ_j for some $1 \leq j \leq r_2$, then $|\mathbf{a}|_v \leq \left(\sum_{j=1}^{wd} x_j^2 \right)^{1/2} \leq \sqrt{wd} |\mathbf{x}|$. Hence for each $v \mid \infty$,

$$(10) \quad |\mathbf{x}| \leq |\mathbf{a}|_v \leq \sqrt{wd} |\mathbf{x}|.$$

Let L_1, \dots, L_t be the linear forms defined in (5). For each $1 \leq i \leq t$, we define

$$|L_i|_v = \max_{1 \leq j \leq wd} |b_{ij}|_v,$$

for each place $v \in M(E)$, and define the height of L_i to be

$$h(L_i) = H(1, b_{i1}, \dots, b_{i(wd)}) = \prod_{v \in M(E)} \max\{1, |L_i|_v\}^{\ell_v/\ell}.$$

We similarly define the height of the matrix B to be

$$h(B) = H(1, b_{11}, \dots, b_{t(wd)}),$$

then $h(L_i) \leq h(B)$ for all $1 \leq i \leq t$. We are now ready to proceed.

3. AN EFFECTIVE VERSION OF KRONECKER'S THEOREM

In this section we derive an effective version of Kronecker's theorem, which we then use to prove Theorems 1.1 and 1.2. Similar to the setup in the beginning of Section 1, let $1, \theta_1, \dots, \theta_t$ be \mathbb{Q} -linearly independent real algebraic numbers. For each $1 \leq j \leq t$, let $f_j(x) \in \mathbb{Z}[x]$ be the minimal polynomial of θ_j of degree d_j , $|f_j|$

be the maximum of absolute values of the coefficients of f_j , and A_j be the leading coefficient of f_j , so $A_j \leq |f_j|$. By Lemma 3.11 of [18],

$$\frac{1}{2^{d_j}} |f_j| \leq h(\theta_j)^{d_j} \leq \sqrt{d_j + 1} |f_j|,$$

for every $1 \leq j \leq t$. Define A to be the least common multiple of A_1, \dots, A_t , so

$$(11) \quad A \leq \prod_{j=1}^t |f_j| \leq \prod_{j=1}^t (2h(\theta_j))^{d_j}.$$

Let $F = \mathbb{Q}(\theta_1, \dots, \theta_t)$ be a number field of degree $e \geq t + 1$, then $e \leq \prod_{j=1}^t d_j$. Let $\theta_{t+1}, \dots, \theta_{e-1} \in F$ be such that

$$1 = \theta_0, \theta_1, \dots, \theta_t, \theta_{t+1}, \dots, \theta_{e-1}$$

form a \mathbb{Q} -basis for F . Let $\sigma_1, \dots, \sigma_e$ be the embeddings of F into \mathbb{C} . We recall Liouville inequality. For any $\mathbf{m} = (m_0, \dots, m_t, 0, \dots, 0) \in \mathbb{Z}^e$,

$$(12) \quad A^e \prod_{i=1}^e \left| \sum_{j=0}^{e-1} \sigma_i(\theta_j) m_j \right| \geq 1,$$

and so

$$(13) \quad A^e \left((t+1) \max_{1 \leq i \leq e, 0 \leq j \leq t} |\sigma_i(\theta_j)| \right)^{e-1} |\mathbf{m}|^{e-1} \|m_1 \theta_1 + \dots + m_t \theta_t\| \geq 1.$$

Now observe that

$$\max_{1 \leq i \leq e, 0 \leq j \leq t} |\sigma_i(\theta_j)| \leq \max_{1 \leq j \leq t} h(\theta_j)^{d_j},$$

and so define

$$(14) \quad \mathcal{C}_1 = \mathcal{C}_1(\theta_1, \dots, \theta_t) := \left((t+1) \max_{1 \leq j \leq t} h(\theta_j)^{d_j} \right)^{e-1} \prod_{j=1}^t (2h(\theta_j))^{ed_j}.$$

Then for any $\mathbf{0} \neq \mathbf{m} \in \mathbb{Z}^t$,

$$(15) \quad \|m_1 \theta_1 + \dots + m_t \theta_t\| \geq \mathcal{C}_1^{-1} |\mathbf{m}|^{-e+1}.$$

We will now apply a transference homogeneous-inhomogeneous argument. A transference principle of this sort was first described in Chapter V, §4 of [3]; the particular stronger result we are applying here is obtained in [2]. Let us write

$$M(\mathbf{y}) = \sum_{i=1}^t \theta_i y_i$$

for $\mathbf{y} = (y_1, \dots, y_t) \in \mathbb{Z}^t$, and let

$$L_j(x) = \theta_j x, \quad 1 \leq j \leq t$$

for $x \in \mathbb{Z}$. Then (15) guarantees that for any $\mathbf{0} \neq \mathbf{y} \in \mathbb{Z}^t$ with $|\mathbf{y}| \leq Y$,

$$\|M(\mathbf{y})\| \geq \mathcal{C}_1^{-1} Y^{-(e-1)}.$$

Now applying the transference Lemma 3 of [2] to these linear forms, we have that for every $\mathbf{a} = (a_1, \dots, a_t) \in \mathbb{R}^t$ there exists $x \in \mathbb{Z}$ such that $|x| \leq 2^{-t}((t+1)!)^2 \mathcal{C}_1 Y^{e-1}$ and

$$\max_{1 \leq j \leq t} \|L_j(x) - a_j\| \leq 2^{-t}((t+1)!)^2 Y^{-1}.$$

Letting $Q = (2^t((t+1)!)^{-2}Y)^{e-1}$, we obtain that

$$\max_{1 \leq j \leq t} \|L_j(x) - \alpha_j\| \leq Q^{-\frac{1}{e-1}}$$

for some $0 \neq x \in \mathbb{Z}$ with $|x| \leq 2^{-et}((t+1)!)^{2e}C_1Q$. Taking $\varepsilon = Q^{-\frac{1}{e-1}}$ immediately yields the following effective version of Kronecker's theorem.

Theorem 3.1. *Let $1, \theta_1, \dots, \theta_t$ be \mathbb{Q} -linearly independent real algebraic numbers, and let $e = [\mathbb{Q}(\theta_1, \dots, \theta_t) : \mathbb{Q}]$. Let C_1 be given by (14) above, and let $\varepsilon > 0$. Then for any $(a_1, \dots, a_t) \in \mathbb{R}^t$ there exists $q \in \mathbb{Z} \setminus \{0\}$ such that*

$$(16) \quad \|q\theta_j - a_j\| \leq \varepsilon, \quad 1 \leq j \leq t$$

and

$$|q| \leq 2^{-et}((t+1)!)^{2e}C_1\varepsilon^{-e+1}.$$

In particular, if $h(\theta_j) \leq H$ for all $1 \leq j \leq t$ and $\max\{e, d_1, \dots, d_t\} \leq \ell$, then

$$|q| \leq \left(2^{\ell t(\ell-1)}(t+1)^{3\ell-1}(t!)^{2\ell}H^{\ell^2(t+1)-\ell}\right)\varepsilon^{-\ell+1}.$$

Remark 3.1. Stronger non-effective results can be derived as corollaries of Schmidt's subspace theorem. For instance, results discussed in Chapter 6, §2 of [16] together with the transference principles of Chapter V, §4 of [3] and [2] imply, for any $\varepsilon > 0$ and $\mathbf{a} \in \mathbb{R}^t$ under the assumptions of Theorem 3.1, the existence of $q \in \mathbb{Z}$ satisfying 16 such that

$$|q| \leq C'(\delta)\varepsilon^{-t-\delta},$$

for any $\delta > 0$, where the constant $C'(\delta)$ is non-effective. This would result in the same exponent on ε in the bounds for $|q|$ in Theorems 1.1 and 1.2, but with non-effective constants.

4. PROOF OF THEOREM 1.1

Here we present the proof of our first result. Since $\Lambda_K(\mathcal{M}) \not\subseteq \mathcal{Z}_{\mathcal{S}}$, $\Lambda_K(\mathcal{M}) \not\subseteq Z(\mathcal{S}_i)$ for all $1 \leq i \leq m$, and so for each i at least one polynomial P_i in \mathcal{S}_i is not identically zero on $\Lambda_K(\mathcal{M})$. Clearly for each $1 \leq i \leq m$,

$$Z(\mathcal{S}_i) \subseteq Z(P_i) := \{\mathbf{x} \in \mathbb{R}^{wd} : P_i(\mathbf{x}) = 0\}.$$

Define

$$P(\mathbf{x}) = \prod_{i=1}^m P_i(\mathbf{x}),$$

so that $\Lambda_K(\mathcal{M}) \not\subseteq Z(P)$ while $\mathcal{Z}_{\mathcal{S}} \subseteq Z(P)$ and $\deg(P) \leq M_{\mathcal{S}}$. We will next construct a point $\mathbf{y} \in \Lambda_K(\mathcal{M})$ of controlled sup-norm such that $P(\mathbf{y}) \neq 0$.

Let $V = \text{span}_{\mathbb{R}} \Lambda_K(\mathcal{M})$ be the sd -dimensional subspace of \mathbb{R}^{wd} spanned by the lattice $\Lambda_K(\mathcal{M})$. For a positive real number μ , let us write

$$C_V(\mu) := \{\mathbf{x} \in V : |\mathbf{x}| \leq \mu\}$$

for the sd -dimensional cube with side-length 2μ centered at the origin in V , so $C_V(\mu) = \mu C_V(1)$. Let $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_{sd}$ be the successive minima of $\Lambda_K(\mathcal{M})$ with respect to the cube $C_V(1)$. In other words, for each $1 \leq i \leq sd$,

$$\lambda_i := \min \{\mu \in \mathbb{R}_{>0} : \dim_{\mathbb{R}} \text{span}_{\mathbb{R}} (\Lambda_K(\mathcal{M}) \cap C_V(\mu)) \geq i\}.$$

Let $\mathbf{v}_1, \dots, \mathbf{v}_{sd}$ be a collection of linearly independent vectors in $\Lambda_K(\mathcal{M})$ corresponding to these successive minima, then $|\mathbf{v}_i| = \lambda_i$. Since the volume of sd -dimensional cube $C_V(1)$ is 2^{sd} , Minkowski's Successive Minima Theorem (see, for instance, [4] or [10]) implies that

$$\frac{\det(\Lambda_K(\mathcal{M}))}{(sd)!} \leq \prod_{i=1}^{sd} |\mathbf{v}_i| \leq \det(\Lambda_K(\mathcal{M})),$$

where $\frac{1}{\sqrt{2}}h(\alpha)^{-1} \leq |\mathbf{v}_1| \leq \dots \leq |\mathbf{v}_{sd}|$, by (9). This means that

$$(17) \quad |\mathbf{v}_1| \leq \dots \leq |\mathbf{v}_{sd}| \leq \left(\sqrt{2}h(\alpha)\right)^{sd-1} \det(\Lambda_K(\mathcal{M})).$$

Let $I(M_S) = \{0, 1, 2, \dots, M_S\}$ be the set of the first $M_S + 1$ non-negative integers. For each $\boldsymbol{\xi} \in I(M_S)^{sd}$, define

$$\mathbf{v}(\boldsymbol{\xi}) = \sum_{i=1}^{sd} \xi_i \mathbf{v}_i,$$

then

$$(18) \quad |\mathbf{v}(\boldsymbol{\xi})| = \max_{1 \leq j \leq wd} \left| \sum_{i=1}^{sd} \xi_i v_{ij} \right| \leq sd |\boldsymbol{\xi}| |\mathbf{v}_{sd}| \leq sd M_S \left(\sqrt{2}h(\alpha)\right)^{sd-1} \det(\Lambda_K(\mathcal{M})),$$

by (17). Assume that $P(\mathbf{v}(\boldsymbol{\xi})) = 0$ for each $\boldsymbol{\xi} \in I(M_S)^{sd}$. Then Theorem 4.2 of [7] implies that $P(\mathbf{x})$ must be identically zero on V , which would contradict the fact that P does not vanish identically on $\Lambda_K(\mathcal{M})$. Hence there must exist some $\boldsymbol{\xi} \in I(M_S)^{sd}$ such that P does not vanish at the corresponding $\mathbf{y} := \mathbf{v}(\boldsymbol{\xi})$, and $|\mathbf{y}| \leq sd M_S \left(\sqrt{2}h(\alpha)\right)^{sd-1} \det(\Lambda_K(\mathcal{M}))$ by (18). Since $P(\mathbf{x})$ is a homogeneous polynomial, it must be true that $P(n\mathbf{y}) \neq 0$ for every $n \in \mathbb{Z}_{>0}$. On the other hand, by our construction

$$n\mathbf{y} = n \sum_{i=1}^{sd} \xi_i \mathbf{v}_i \in \text{span}_{\mathbb{Z}} \{\mathbf{v}_1, \dots, \mathbf{v}_{sd}\} \subseteq \Lambda_K(\mathcal{M}),$$

and so $\{n\mathbf{y}\}_{n \in \mathbb{Z}_{>0}}$ gives an infinite sequence of points in $\Lambda_K(\mathcal{M})$ outside of \mathcal{Z}_S . For each such point, we have

$$L_i(n\mathbf{y}) = nL_i(\mathbf{y}), \quad \forall 1 \leq i \leq t.$$

Let us define, for each $1 \leq i \leq t$,

$$(19) \quad \theta_i := L_i(\mathbf{y}) = \sum_{j=1}^{wd} b_{ij} y_j \neq 0,$$

since $y_j \in K_1$, not all zero, and b_{ij} are K_1 -linearly independent. Notice that $\theta_1, \dots, \theta_t \in E$, and hence all of them are algebraic numbers of degree $\leq \ell$.

Let $\alpha \in \mathfrak{U}_K(\mathcal{M})$. Then, by (10), for each archimedean $v \in M(E)$,

$$(20) \quad \begin{aligned} \max\{1, |\theta_i|_v\} &\leq \max\{1, (wd)^{\frac{3}{2}} |L_i|_v |\mathbf{y}|\} \leq (wd)^{\frac{3}{2}} \max\{1, |\mathbf{y}|\} \max\{1, |L_i|_v\} \\ &\leq \sqrt{2} (wd)^{\frac{3}{2}} h(\alpha) |\mathbf{y}| \max\{1, |L_i|_v\}, \end{aligned}$$

by (9). By (18), $|\mathbf{y}| \leq sd M_S \left(\sqrt{2}h(\alpha)\right)^{sd-1} \det(\Lambda_K(\mathcal{M}))$, and hence

$$(21) \quad \max\{1, |\theta_i|_v\} \leq sd (wd)^{\frac{3}{2}} M_S \left(\sqrt{2}h(\alpha)\right)^{sd} \det(\Lambda_K(\mathcal{M})) \max\{1, |L_i|_v\}.$$

Now suppose $v \in M(E)$ is non-archimedean. Then αy_j is an algebraic integer for each $1 \leq j \leq wd$, and hence $|\alpha y_j|_v = |\alpha|_v |y_j|_v \leq 1$, meaning that

$$\max\{1, |y_1|_v, \dots, |y_{wd}|_v\} \leq \max\{1, |\alpha|_v^{-1}\}.$$

Then

$$\begin{aligned} \max\{1, |\theta_i|_v\} &\leq \max\{1, |L_i|_v\} \max\{1, |y_1|_v, \dots, |y_{wd}|_v\} \\ (22) \quad &\leq \max\{1, |\alpha|_v^{-1}\} \max\{1, |L_i|_v\}, \end{aligned}$$

for each non-archimedean $v \in M(E)$. Taking a product over all places of E , we obtain:

$$\begin{aligned} h(\theta_i) &= \prod_{v \in M(E)} \max\{1, |\theta_i|_v\}^{\frac{\ell_v}{\ell}} = \left(\prod_{v|\infty} \max\{1, |\theta_i|_v\}^{\ell_v} \times \prod_{v \nmid \infty} \max\{1, |\theta_i|_v\}^{\ell_v} \right)^{\frac{1}{\ell}} \\ &\leq sd(wd)^{\frac{3}{2}} M_S \left(\sqrt{2} h(\alpha) \right)^{sd} \det(\Lambda_K(\mathcal{M})) h(L_i) \prod_{v \nmid \infty} \max\{1, |\alpha|_v^{-1}\}^{\frac{\ell_v}{\ell}} \\ &\leq sd(wd)^{\frac{3}{2}} M_S \left(\sqrt{2} h(\alpha) \right)^{sd} h(\alpha^{-1}) \det(\Lambda_K(\mathcal{M})) h(L_i). \end{aligned}$$

Recalling that $h(L_i) \leq h(B)$ for all $1 \leq i \leq t$, we obtain

$$(23) \quad h(\theta_i) \leq 2^{\frac{sd}{2}} sd(wd)^{\frac{3}{2}} M_S h(\alpha)^{sd} h(\alpha^{-1}) \det(\Lambda_K(\mathcal{M})) h(B),$$

for each $1 \leq i \leq t$, where the choice of $\alpha \in \mathfrak{U}_K(\mathcal{M})$ is arbitrary.

We will now show that $1, \theta_1, \dots, \theta_t$ are \mathbb{Q} -linearly independent. Suppose not, then there exist $c_0, c_1, \dots, c_t \in \mathbb{Q}$, not all zero, such that

$$c_0 = \sum_{i=1}^t c_i \theta_i = \sum_{i=1}^t \sum_{j=1}^{wd} c_i y_j b_{ij},$$

where not all $c_i y_j$ are equal to zero. Recall that $\mathbf{y} \in \Lambda_K(\mathcal{M})$, meaning that coordinates of \mathbf{y} are in K_1 , hence all $c_i y_j$ are in K_1 . This contradicts the assumption that $1, b_{11}, \dots, b_{1(wd)}$ are linearly independent over K_1 . Hence $1, \theta_1, \dots, \theta_t$ must be linearly independent over \mathbb{Q} .

Now let $\mathbf{a} = (a_1, \dots, a_t) \in \mathbb{R}^t$ and $\varepsilon > 0$, as in the statement of our theorem. Then, by (23) and Theorem 3.1, there exists $q \in \mathbb{Z}$ and $\mathbf{p} \in \mathbb{Z}^t$ such that

$$\begin{aligned} |q| &\leq 2^{\ell t(\ell-1)} (t+1)^{3\ell-1} (t!)^{2\ell} \times \\ (24) \quad &\times \left(2^{\frac{sd}{2}} sd(wd)^{\frac{3}{2}} M_S h(\alpha)^{sd} h(\alpha^{-1}) \det(\Lambda_K(\mathcal{M})) h(B) \right)^{\ell^2(t+1)-\ell} \varepsilon^{-\ell+1} \end{aligned}$$

and

$$|q\theta_i - a_i - p_i| < \varepsilon \quad \forall 1 \leq i \leq t.$$

Letting $\mathbf{x} = q\mathbf{y}$, we see that $q\theta_i = L_i(\mathbf{x})$ for each $1 \leq i \leq t$ and $|\mathbf{x}| = |q||\mathbf{y}|$. Combining these observations with (18), (24) and (8) and taking a minimum over all $\alpha \in \mathfrak{U}_K(\mathcal{M})$ finishes the proof of the theorem.

5. PROOF OF THEOREM 1.2

Let $\Gamma_1, \dots, \Gamma_m$ be full-rank sublattices of $\Lambda_K(\mathcal{M})$ of respective determinants $\mathcal{D}_1, \dots, \mathcal{D}_m$. Let $\Omega = \cap_{i=1}^m \Gamma_i$, then Ω also has full rank and

$$\mathcal{D} := \mathcal{D}_1 \cdots \mathcal{D}_m \geq \det \Omega.$$

We write λ_i for the successive minima of $\Lambda_K(\mathcal{M})$ and $\lambda_i(\Omega)$ for the successive minima of Ω . Theorem 1.2 of [12] implies that there exists $\mathbf{y} \in \Lambda_K(\mathcal{M}) \setminus \bigcup_{i=1}^m \Gamma_i$ such that

$$|\mathbf{y}| < \frac{\det \Lambda_K(\mathcal{M})}{\lambda_1(\Omega)^{sd-1}} \left(\sum_{i=1}^m \frac{\mathcal{D}}{\mathcal{D}_i} - m + 1 \right) + \lambda_1(\Omega).$$

Our first goal is to make this bound more explicit in terms of the parameters of \mathcal{M} . First notice that by Minkowski's Successive Minima Theorem,

$$\lambda_1(\Omega) \leq \left(\prod_{i=1}^{sd} \lambda_i(\Omega) \right)^{1/sd} \leq (\det \Omega)^{1/sd} \leq \mathcal{D}^{1/sd}.$$

We also need a lower bound on $\lambda_1(\Omega)$. Observe that $\lambda_1(\Omega) \geq \lambda_1$, while $\lambda_1 \geq \frac{1}{\sqrt{2}} h(\alpha)^{-1}$ for any $\alpha \in \mathfrak{U}_K(\mathcal{M})$, by (9) above. Putting these estimates together, we see that

$$(25) \quad |\mathbf{y}| < \left(\sqrt{2} h(\alpha) \right)^{sd-1} \det \Lambda_K(\mathcal{M}) \left(\sum_{i=1}^m \frac{\mathcal{D}}{\mathcal{D}_i} - m + 1 \right) + \mathcal{D}^{1/sd}$$

for any $\alpha \in \mathfrak{U}_K(\mathcal{M})$.

Since $\mathbf{y} \in \Lambda_K(\mathcal{M})$ and $|\Lambda_K(\mathcal{M}) : \Gamma_i| = \mathcal{D}_i / \det \Lambda_K(\mathcal{M})$ for each $1 \leq i \leq m$, it follows that

$$(g|\Lambda_K(\mathcal{M}) : \Gamma_i|) \mathbf{y} = \frac{g\mathcal{D}_i}{\det \Lambda_K(\mathcal{M})} \mathbf{y} \in \Gamma_i,$$

for every $g \in \mathbb{Z}$, and hence

$$\left(\frac{g\mathcal{D}_1 \cdots \mathcal{D}_m}{(\det \Lambda_K(\mathcal{M}))^m} \right) \mathbf{y} = \left(\frac{g\mathcal{D}}{(\det \Lambda_K(\mathcal{M}))^m} \right) \mathbf{y} \in \Omega,$$

for every $g \in \mathbb{Z}$. Therefore, it must be true that

$$\left(\frac{g\mathcal{D}}{(\det \Lambda_K(\mathcal{M}))^m} + 1 \right) \mathbf{y} \in \Lambda_K(\mathcal{M}) \setminus \bigcup_{i=1}^m \Gamma_i,$$

for every $g \in \mathbb{Z}$. For brevity, let us write $\mathcal{D}' = \frac{\mathcal{D}}{(\det \Lambda_K(\mathcal{M}))^m}$.

From here on, the argument is largely similar to the proof of Theorem 1.1 above, but with some notable changes. For each $1 \leq i \leq t$, let θ_i be as in (19) for our choice of $\mathbf{y} \in \Lambda_K(\mathcal{M}) \setminus \bigcup_{i=1}^m \Gamma_i$ satisfying (25) as above, then

$$L_i((g\mathcal{D}' + 1)\mathbf{y}) = (g\mathcal{D}' + 1)\theta_i \quad \forall 1 \leq i \leq t.$$

Using (20) with (25) instead of (18), we obtain that $\max\{1, |\theta_i|_v\} \leq$

$$(wd)^{\frac{3}{2}} \left(\left(\sqrt{2} h(\alpha) \right)^{sd} \det \Lambda_K(\mathcal{M}) \left(\sum_{i=1}^m \frac{\mathcal{D}}{\mathcal{D}_i} - m + 1 \right) + \mathcal{D}^{\frac{1}{sd}} \sqrt{2} h(\alpha) \right) \max\{1, |L_i|_v\}$$

for all archimedean $v \in M(E)$, while for the non-archimedean $v \in M(E)$,

$$\max\{1, |\theta_i|_v\} \leq \max\{1, |\alpha^{-1}|_v\} \max\{1, |L_i|_v\},$$

as in (22). Taking the product over all places of E , we have for every $1 \leq i \leq t$:

$$(26) \quad \begin{aligned} h(\theta_i) &\leq (wd)^{\frac{3}{2}} \sqrt{2} h(\alpha) h(\alpha^{-1}) h(B) \times \\ &\times \left(\left(\sqrt{2} h(\alpha) \right)^{sd-1} \det \Lambda_K(\mathcal{M}) \left(\sum_{i=1}^m \frac{\mathcal{D}}{\mathcal{D}_i} - m + 1 \right) + \mathcal{D}^{\frac{1}{sd}} \right), \end{aligned}$$

and $1, \theta_1, \dots, \theta_t$ (and hence $1, \mathcal{D}'\theta_1, \dots, \mathcal{D}'\theta_t$) are \mathbb{Q} -linearly independent by the same reasoning as in the proof of Theorem 1.1.

Now let $\mathbf{a} = (a_1, \dots, a_t) \in \mathbb{R}^t$ and $\varepsilon > 0$, as in the statement of our theorem. Notice that for each $1 \leq i \leq t$,

$$|(g\mathcal{D}' + 1)\theta_i - a_i - p_i| = |g(\mathcal{D}'\theta_i) + (\theta_i - a_i) - p_i|,$$

for any integers p_1, \dots, p_t . Then, applying Theorem 3.1 to approximate the vector $(\theta_1 - a_1, \dots, \theta_t - a_t)$ by the fractional parts of the integer multiples of the vector $(\mathcal{D}'\theta_1, \dots, \mathcal{D}'\theta_t)$, we conclude that there exists $g \in \mathbb{Z}$ and $\mathbf{p} \in \mathbb{Z}^t$ such that

$$(27) \quad \begin{aligned} |g| &\leq 2^{\ell t(\ell-1)} (t+1)^{3\ell-1} (t!)^{2\ell} \times \\ &\times \left((wd)^{\frac{3}{2}} \sqrt{2} h(\alpha) h(\alpha^{-1}) h(B) \mathcal{E}_\alpha(\mathcal{M}, \Gamma_1, \dots, \Gamma_m) \right)^{\ell^2(t+1)-\ell} \varepsilon^{-\ell+1}, \end{aligned}$$

where $\mathcal{E}_\alpha(\mathcal{M}, \Gamma_1, \dots, \Gamma_m)$ is as in (6), and

$$|g(\mathcal{D}'\theta_i) + (\theta_i - a_i) - p_i| < \varepsilon \quad \forall 1 \leq i \leq t.$$

Letting $\mathbf{x} = (g\mathcal{D}' + 1)\mathbf{y}$, we see that $(g\mathcal{D}' + 1)\theta_i = L_i(\mathbf{x})$ for each $1 \leq i \leq t$ and $|\mathbf{x}| = |g\mathcal{D}' + 1||\mathbf{y}|$. Combining these observations with (25), (27) and (8) finishes the proof of the theorem.

Acknowledgement. The author thanks Joshua Corwin for several helpful conversations on the subject of this paper, as well as Gregorio Malajovich and Vesselin Dimitrov for useful communications. The author is especially grateful to the referee for extremely helpful suggestions, which have substantially improved the quality of the bounds; in particular, Section 3 of the paper is due to the referee.

REFERENCES

- [1] N. Alon. Combinatorial nullstellensatz. *Combin. Probab. Comput.*, 8(1-2):7–29, 1999.
- [2] Y. Bugeaud and M. Laurent. On exponents of homogeneous and inhomogeneous Diophantine approximation. *Mosc. Math. J.*, 5(4):747–766, 972, 2005.
- [3] J. W. S. Cassels. *An introduction to Diophantine approximations*. Cambridge Univ. Press, 1957.
- [4] J. W. S. Cassels. *An Introduction to the Geometry of Numbers*. Springer-Verlag, 1997.
- [5] L. G. P. Dirichlet. Verallgemeinerung eines Satzes aus der Lehre von den Kettenbrüchen nebst einigen Anwendungen auf die Theorie der Zahlen. *S. B. Preuss. Akad. Wiss.*, pages 93–95, 1842.
- [6] M. Drmota and R. F. Tichy. *Sequences, discrepancies and applications. Lecture Notes in Mathematics, 1651*. Springer-Verlag, Berlin, 1997.
- [7] L. Fukshansky. Algebraic points of small height missing a union of varieties. *J. Number Theory*, 130(10):2099–2118, 2010.
- [8] L. Fukshansky and G. Henshaw. Lattice point counting and height bounds over number fields and quaternion algebras. *Online J. Anal. Comb.*, 8(5):20 pp., 2013.
- [9] S. M. Gonek and H. L. Montgomery. Kronecker's approximation theorem. *preprint*, 2015.
- [10] P. M. Gruber and C. G. Lekkerkerker. *Geometry of Numbers*. Second edition. North-Holland Mathematical Library, 37. North-Holland Publishing Co., Amsterdam, 1987.

- [11] G. H. Hardy and E. M. Wright. *An introduction to the theory of numbers. Fifth edition.* The Clarendon Press, Oxford University Press, New York,, 1979.
- [12] M. Henk and C. Thiel. Restricted successive minima. *Pacific J. Math.*, 269(2):341–354, 2014.
- [13] L. Kronecker. Näherungsweise ganzzahlige Auflösung linearer Gleichungen. *Monatsber. Königlich. Preuss. Akad. Wiss. Berlin*, pages 1179–1193, 1271–1299, 1884.
- [14] S. Lang. *Algebraic Number Theory.* Springer-Verlag, 1994.
- [15] G. Malojuvich. An effective version of Kronecker’s theorem on simultaneous Diophantine approximation. Technical report, City University of Hong Kong, 1996. <http://www.labma.ufrj.br/~gregorio/papers/kron.pdf>.
- [16] W. M. Schmidt. *Diophantine Approximation.* Springer-Verlag, 1980.
- [17] T. Vorselen. On Kronecker’s theorem over the adèles. Master’s thesis, Universiteit Leiden, 4 2010.
- [18] M. Waldschmidt. *Diophantine approximation on linear algebraic groups.* Springer-Verlag, Berlin, 2000.

DEPARTMENT OF MATHEMATICS, 850 COLUMBIA AVENUE, CLAREMONT MCKENNA COLLEGE,
CLAREMONT, CA 91711

E-mail address: `lenny@cmc.edu`